

# Proper Inclusions of Morrey Spaces

Hendra Gunawan, Denny I. Hakim, and Mochammad Idris

Department of Mathematics, Bandung Institute of Technology  
Bandung 40132, Indonesia

## Abstract

In this paper, we prove that the inclusions between Morrey spaces, between weak Morrey spaces, and between a Morrey space and a weak Morrey space are all proper. The proper inclusion between a Morrey space and a weak Morrey space is established via the unboundedness of the Hardy-Littlewood maximal operator on Morrey spaces of exponent 1. In addition, we also give a necessary condition for each inclusion. Our results refine previous inclusion properties studied in [3].

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**Keywords:** Morrey spaces, weak Morrey spaces, inclusion properties.

## 1 Introduction

Morrey spaces were first introduced by C.B. Morrey in [6] in relation to the study of the solution of certain elliptic partial differential equations. For  $1 \leq p \leq q < \infty$ , the *Morrey space*  $\mathcal{M}_q^p = \mathcal{M}_q^p(\mathbb{R}^d)$  is defined to be the set of all  $f \in L_{\text{loc}}^p(\mathbb{R}^d)$  such that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{a \in \mathbb{R}^d, r > 0} |B(a, r)|^{\frac{1}{q}} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Here,  $B(a, r)$  is an open ball centered at  $a$  with radius  $r$ , and  $|B(a, r)|$  denotes its Lebesgue measure. Notice that, when  $p = q$ , one can recover the Lebesgue space  $L^p(\mathbb{R}^d)$  as the special case of  $\mathcal{M}_q^p(\mathbb{R}^d)$ . See [9] for various spaces related to Morrey spaces. Many researchers have proved the boundedness of classical integral operators on Morrey spaces and their generalizations. See, for instance, [1, 2, 7].

Concerning the Hardy-Littlewood maximal operator (defined in Section 3), one may prove its boundedness on Morrey spaces using the inclusion  $\mathcal{M}_q^p \subseteq \mathcal{M}_q^1$ . In general, we have the following inclusions of Morrey spaces

$$L^q(\mathbb{R}^d) = \mathcal{M}_q^q(\mathbb{R}^d) \subseteq \mathcal{M}_q^{p_2}(\mathbb{R}^d) \subseteq \mathcal{M}_q^{p_1}(\mathbb{R}^d)$$

provided that  $1 \leq p_1 \leq p_2 \leq q$  [12]. These inclusions may be obtained by applying Hölder's inequality. Note that, for  $1 \leq p_2 < q$ , we have  $f(x) := |x|^{-d/q} \in \mathcal{M}_q^{p_2} \setminus \mathcal{M}_q^q$ . This tells us that the inclusion  $\mathcal{M}_q^q(\mathbb{R}^d) \subseteq \mathcal{M}_q^{p_2}(\mathbb{R}^d)$  is proper for  $1 \leq p_2 < q$ . One may ask further

whether the inclusion  $\mathcal{M}_q^{p_2}(\mathbb{R}^d) \subseteq \mathcal{M}_q^{p_1}(\mathbb{R}^d)$  is proper for  $1 \leq p_1 < p_2 < q$ . In this paper, we shall give the affirmative answer to this question by constructing a function in  $\mathcal{M}_q^{p_1} \setminus \mathcal{M}_q^{p_2}$  for  $1 \leq p_1 < p_2 < q$ .

Besides the ‘strong’ Morrey spaces, we also have weak Morrey spaces whose definitions are given as follows:

**Definition 1.1.** Let  $1 \leq p \leq q < \infty$ . A measurable functions  $f$  on  $\mathbb{R}^d$  is said to belong to the *weak Morrey space*  $w\mathcal{M}_q^p = w\mathcal{M}_q^p(\mathbb{R}^d)$  if the quasi-norm

$$\|f\|_{w\mathcal{M}_q^p} := \sup_{\gamma > 0} \|\gamma \chi_{\{|f| > \gamma\}}\|_{\mathcal{M}_q^p}$$

is finite.

Note that, by using the inequality  $\gamma \chi_{\{|f| > \gamma\}} \leq |f|$  for every  $\gamma > 0$ , we have  $\mathcal{M}_q^p \subseteq w\mathcal{M}_q^p$ . The inclusion properties of weak Morrey spaces, generalized Morrey spaces, generalized weak Morrey spaces, and their necessary conditions were discussed in [3]. In particular, for the case of Morrey spaces and weak Morrey spaces, the results can be stated as follows:

**Theorem 1.2.** [3] *For  $1 \leq p_1 \leq p_2 \leq q < \infty$ , the following inclusion holds:*

$$w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}.$$

*Further, if  $p_1 < p_2$ , then*

$$w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}.$$

In addition to the above inclusion relations of Morrey spaces, we have the following theorems.

**Theorem 1.3.** *Let  $1 \leq p_1 < p_2 < q$ . Then each of the following inclusions is proper:*

- (i)  $\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$ ;
- (ii)  $w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$ ;
- (iii)  $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$ .

**Theorem 1.4.** *Let  $1 \leq p \leq q$ . Then the inclusion  $\mathcal{M}_q^p \subseteq w\mathcal{M}_q^p$  is proper.*

We also obtain the following necessary conditions for inclusion of Morrey spaces and weak Morrey spaces which can be seen as a refinement of some necessary conditions given in [3].

**Theorem 1.5.** *Let  $1 \leq p_1 \leq q_1 < \infty$  and  $1 \leq p_2 \leq q_2 < \infty$ . Then the following implications hold:*

- (i)  $\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$  implies  $q_1 = q_2$  and  $p_1 \leq p_2$ ;
- (ii)  $w\mathcal{M}_{q_2}^{p_2} \subseteq w\mathcal{M}_{q_1}^{p_1}$  implies  $q_1 = q_2$  and  $p_1 \leq p_2$ ;
- (iii)  $w\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$  implies  $q_1 = q_2$  and  $p_1 < p_2$ .

*Remark 1.6.* A necessary and sufficient condition for inclusion of Morrey spaces on a bounded domain can be found in [10, Theorem 2.1]. The case of Morrey spaces on  $\mathbb{R}^d$  is mentioned in [5, Eq. (3.9)] and the authors refer to [11, Satz 1.6]. However, we do not have the access to the paper, so that we do not know how the proof goes. Here we present a proof of the necessary and sufficient condition for the inclusion property, which is different from and simpler than that in [10].

The organization of this paper is as follows. In the next section, we prove that for  $1 \leq p_1 < p_2 < q$  the set  $\mathcal{M}_q^{p_1} \setminus \mathcal{M}_q^{p_2}$  is not empty. By the same example, we also show that for  $1 \leq p_1 < p_2 < q$  the inclusion  $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$  is proper. In Section 3, we give the proof of Theorem 1.4 using the unboundedness of the Hardy-Littlewood maximal operator on Morrey spaces of exponent 1. The proof of Theorem 1.5 is given in the last section. Throughout this paper, we denote by  $C$  a positive constant which is independent of the function  $f$  and its value may be different from line to line.

## 2 The proof of Theorem 1.3

We shall first prove Theorem 1.3 (i) by constructing a function which belongs to  $\mathcal{M}_q^{p_1}$  but not to  $\mathcal{M}_q^{p_2}$ , for  $1 \leq p_1 < p_2 < q$ .

*Proof of Theorem 1.3 (i).* Let  $1 \leq p_1 < p_2 < q$  and  $\beta := \frac{d(p_1+p_2)}{2q}$ . Then we have

$$\frac{dp_1}{q} < \beta < \frac{dp_2}{q} \quad (2.1)$$

and

$$d - \beta = \frac{d(q - p_1) + d(q - p_2)}{2q} > 0.$$

Define  $g(x) := \chi_{B(0,1)}(x) + \chi_{\mathbb{R}^n \setminus B(0,1)}(x)|x|^{-\beta}$ . Then, for each  $k \in \mathbb{N}$ , we choose  $r_k \in (k, k+1)$  such that

$$\int_{B(0,k+1) \setminus B(0,k)} g(x) \, dx = |B(0, r_k) \setminus B(0, k)|$$

Next define

$$f(x) := \chi_{B(0,1)}(x) + \sum_{k=1}^{\infty} \chi_{B(0,r_k) \setminus B(0,k)}(x). \quad (2.2)$$

We shall show that  $f \in \mathcal{M}_q^{p_1} \setminus \mathcal{M}_q^{p_2}$ . First observe that

$$\int_{B(a,r)} |f(x)|^p \, dx \leq \int_{B(0,r)} |f(x)|^p \, dx$$

for every  $1 \leq p < \infty$ ,  $a \in \mathbb{R}^n$ ,  $r > 0$ . Now, for  $1 \leq p < \infty$  and  $r > 2$ , we have

$$\int_{B(0,r)} |f(x)|^p \, dx = \int_{B(0,r)} |f(x)| \, dx \leq \int_{B(0,2r)} g(x) \, dx,$$

so

$$\int_{B(0,r)} |f(x)|^p dx \leq \int_{B(0,2r)} |x|^{-\beta} dx = Cr^{d-\beta} \quad (2.3)$$

and

$$\int_{B(0,r)} |f(x)|^p dx \geq \int_{B(0,r) \setminus B(0,1)} |x|^{-\beta} dx = C(r^{d-\beta} - 1) \geq C \left(1 - \frac{1}{2^{d-\beta}}\right) r^{d-\beta}. \quad (2.4)$$

Therefore, by substituting  $p = p_1$  into (2.3) and recalling (2.1), we have

$$|B(0,r)|^{\frac{1}{q} - \frac{1}{p_1}} \left( \int_{B(0,r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \leq Cr^{\frac{d}{q} - \frac{d}{p_1}} r^{\frac{d}{p_1} - \frac{\beta}{p_1}} = Cr^{\frac{d}{q} - \frac{\beta}{p_1}} \leq C. \quad (2.5)$$

On the other hand, for each  $r \leq 2$ , we have

$$|B(0,r)|^{\frac{1}{q} - \frac{1}{p_1}} \left( \int_{B(0,r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \leq Cr^{\frac{d}{q} - \frac{d}{p_1}} \left( \int_{B(0,r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \leq Cr^{\frac{d}{q}} \leq C. \quad (2.6)$$

By combining (2.5) and (2.6) we conclude that  $f \in \mathcal{M}_q^{p_1}$ .

Meanwhile, by substituting  $p = p_2$  into (2.4), we have

$$|B(0,r)|^{\frac{1}{q} - \frac{1}{p_2}} \left( \int_{B(0,r)} |f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \geq Cr^{\frac{d}{q} - \frac{d}{p_2}} r^{\frac{d-\beta}{p_2}} = Cr^{\frac{d}{q} - \frac{\beta}{p_2}}.$$

Since  $\frac{d}{q} - \frac{\beta}{p_2} > 0$ , we have

$$\begin{aligned} \sup_{r>0} |B(a,r)|^{\frac{1}{q} - \frac{1}{p_2}} \left( \int_{B(a,r)} |f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} &\geq C \sup_{r>2} |B(0,r)|^{\frac{1}{q} - \frac{1}{p_2}} \left( \int_{B(0,r)} |f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \\ &\geq C \sup_{r>2} r^{\frac{d}{q} - \frac{\beta}{p_2}} = \infty. \end{aligned}$$

Thus  $f \notin \mathcal{M}_q^{p_2}$ , and we are done.  $\square$

Theorem 1.3 (ii) and (iii) are proved by using the function  $f$  in the proof of Theorem 1.3 (i) and its relation with the characteristic function on its level set. The detailed proof goes as follows:

*Proof of Theorem 1.3 (ii)-(iii).* For  $1 \leq p_1 < p_2 < q$ , let  $f$  be defined by (2.2). Observe that

$$\chi_{\{|f|>\gamma\}} = \begin{cases} 0, & \gamma \geq 1, \\ f, & \gamma \in (0, 1). \end{cases}$$

This together with the fact that  $f \notin \mathcal{M}_q^{p_2}$  gives

$$\|f\|_{w\mathcal{M}_q^{p_2}} = \sup_{\gamma \in (0,1)} \gamma \|\chi_{\{|f|>\gamma\}}\|_{\mathcal{M}_q^{p_2}} = \sup_{\gamma \in (0,1)} \gamma \|f\|_{\mathcal{M}_q^{p_2}} = \|f\|_{\mathcal{M}_q^{p_2}} = \infty,$$

and hence  $f \in \mathcal{M}_q^{p_1} \setminus w\mathcal{M}_q^{p_2}$ . Thus we have shown that  $w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$  is a proper inclusion. Since  $\mathcal{M}_q^{p_1} \subseteq w\mathcal{M}_q^{p_1}$ , we also have  $f \in w\mathcal{M}_q^{p_1} \setminus w\mathcal{M}_q^{p_2}$ , so the inclusion (iii) is proper.  $\square$

### 3 The proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemma.

**Lemma 3.1.** *Let  $1 \leq p \leq q < \infty$ . Then*

$$\|f\|_{\mathcal{M}_q^p} = \| |f|^p \|_{\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}}$$

for every  $f \in \mathcal{M}_q^p$  and

$$\|f\|_{w\mathcal{M}_q^p} = \| |f|^p \|_{w\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}}$$

for every  $f \in w\mathcal{M}_q^p$ .

*Proof.* We calculate

$$\|f\|_{\mathcal{M}_q^p} = \sup_B \left( |B|^{\frac{q}{p}-1} \int_B |f(x)|^p dx \right)^{\frac{1}{p}} = \| |f|^p \|_{\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}}.$$

By applying the first identity for  $\chi_{\{|f|>\gamma^{1/p}\}}$ , we have

$$\| |f|^p \|_{w\mathcal{M}_{\frac{q}{p}}^1}^{1/p} = \sup_{\gamma>0} \gamma^{1/p} \| \chi_{\{|f|^p>\gamma\}} \|_{\mathcal{M}_{q/p}^1}^{\frac{1}{p}} = \sup_{\gamma>0} \gamma^{1/p} \| \chi_{\{|f|>\gamma^{1/p}\}} \|_{\mathcal{M}_q^p} = \|f\|_{w\mathcal{M}_q^p},$$

as desired.  $\square$

We also use the following fact about the unboundedness of the Hardy-Littlewood maximal operator  $M$  on Morrey spaces of exponent 1. The operator  $M$  maps a locally integrable function  $f$  to  $Mf$  which is given by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^d.$$

**Lemma 3.2.** *The Hardy-Littlewood maximal operator  $M$  is not bounded on the Morrey space  $\mathcal{M}_q^1$  for  $1 < q < \infty$ .*

*Remark 3.3.* Lemma 3.2 is a consequence of a necessary condition of the boundedness of  $M$  on generalized Orlicz-Morrey spaces given in [8, Corollary 5.3]. The Morrey space  $\mathcal{M}_q^p$  in this paper is recognized as the Orlicz-Morrey space  $L^{(\Phi,\phi)}$  with  $\Phi(t) = t^p$  and  $\phi(t) = t^{-1/q}$ . Based on [8, Corollary 5.3], the maximal operator  $M$  is bounded on  $L^{(\Phi,\phi)}$  if and only if  $\Phi \in \nabla_2$  (that is,  $\Phi(r) \leq \frac{1}{2k} \Phi(kr)$  for some  $k \geq 1$ ). Clearly  $\Phi(t) = t^p \notin \nabla_2$ .

Now, we are ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* Let  $1 \leq p \leq q$ . If  $p = q$ , then  $f(x) := |x|^{-d/q} \in w\mathcal{M}_q^p \setminus \mathcal{M}_q^p$ . So assume that  $p < q$  and write  $r = \frac{q}{p}$ . Suppose to the contrary that  $\mathcal{M}_r^1 = w\mathcal{M}_r^1$ . Then, for every  $g \in \mathcal{M}_r^1$ , we have

$$\|Mg\|_{\mathcal{M}_r^1} = \|Mg\|_{w\mathcal{M}_r^1} \leq C \|g\|_{\mathcal{M}_r^1},$$

so the Hardy-Littlewood maximal operator is bounded on  $\mathcal{M}_r^1$ . This contradicts Lemma 3.2. Therefore,  $w\mathcal{M}_r^1 \setminus \mathcal{M}_r^1 \neq \emptyset$ . We can now choose  $g_0 \in w\mathcal{M}_r^1 \setminus \mathcal{M}_r^1$  and define  $f_0 := g_0^{1/p}$ . Then, by virtue of Lemma 3.1, we have

$$\|f_0\|_{\mathcal{M}_q^p} = \|g_0^{1/p}\|_{\mathcal{M}_q^p} = \|g_0\|_{\mathcal{M}_r^1} = \infty$$

and

$$\|f_0\|_{w\mathcal{M}_q^p} = \|g_0^{1/p}\|_{w\mathcal{M}_q^p} = \|g_0\|_{w\mathcal{M}_r^1} < \infty.$$

Hence  $f_0 \in w\mathcal{M}_q^p \setminus \mathcal{M}_q^p$  as desired.  $\square$

To conclude this section, we write a proposition which gives us a subset of weak Morrey spaces with norm equivalence between the Morrey norm  $\|\cdot\|_{\mathcal{M}_q^p}$  and the weak Morrey quasi-norm  $\|\cdot\|_{w\mathcal{M}_q^p}$ .

**Proposition 3.4.** *Let  $1 \leq p < q < \infty$ . Suppose that  $f$  is a positive radial decreasing function in  $w\mathcal{M}_q^p(\mathbb{R}^d)$ . Then  $f \in \mathcal{M}_q^p(\mathbb{R}^d)$  with*

$$\|f\|_{w\mathcal{M}_q^p} \leq \|f\|_{\mathcal{M}_q^p} \leq \left( \frac{q\omega_{d-1}}{d(q-p)|B(0,1)|} \right)^{1/p} \|f\|_{w\mathcal{M}_q^p},$$

that is,  $\|f\|_{w\mathcal{M}_q^p} \sim \|f\|_{\mathcal{M}_q^p}$ .

*Proof.* Recall that, since  $\gamma\chi_{\{|f|>\gamma\}} \leq |f|$  for every  $\gamma > 0$ , we have  $\|f\|_{w\mathcal{M}_q^p} \leq \|f\|_{\mathcal{M}_q^p}$ . Next, let  $x \in \mathbb{R}^d$ . Since  $\{y \in B(0, |x|) : f(y) > f(x)\} = B(0, |x|)$ , we have

$$\begin{aligned} f(x) &= \frac{f(x)|\{y \in B(0, |x|) : f(y) > f(x)\}|^{1/p}}{|B(0, |x|)|^{1/p}} \\ &\leq \frac{|B(0, |x|)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{w\mathcal{M}_q^p}}{|B(0, |x|)|^{1/p}} \\ &= |B(0, 1)|^{-\frac{1}{q}}\|f\|_{w\mathcal{M}_q^p}|x|^{-d/q}. \end{aligned}$$

By combining the last estimate and

$$\||x|^{-d/q}\|_{\mathcal{M}_q^p} = |B(0, 1)|^{1/q} \left( \frac{q\omega_{d-1}}{d(q-p)|B(0, 1)|} \right)^{1/p},$$

where  $\omega_{d-1}$  is the surface area of the unit sphere  $\mathbb{S}^{d-1}$ , we get

$$\|f\|_{\mathcal{M}_q^p} \leq (|B(0, 1)|^{-\frac{1}{q}}\||x|^{-d/q}\|_{\mathcal{M}_q^p})\|f\|_{w\mathcal{M}_q^p} = \left( \frac{q\omega_{d-1}}{d(q-p)|B(0, 1)|} \right)^{1/p} \|f\|_{w\mathcal{M}_q^p}.$$

Hence  $\|f\|_{w\mathcal{M}_q^p} \sim \|f\|_{\mathcal{M}_q^p}$ .  $\square$

## 4 The proof of Theorems 1.5

*Proof of Theorem 1.5 (i).* It follows from the inclusion  $\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$  that

$$\|\chi_{B(0,r)}\|_{\mathcal{M}_{q_1}^{p_1}} \leq C \|\chi_{B(0,r)}\|_{\mathcal{M}_{q_2}^{p_2}},$$

for every  $r > 0$ . Therefore

$$r^{\frac{d}{q_1} - \frac{d}{q_2}} \leq C$$

for every  $r > 0$ , which implies that  $q_1 = q_2$ . Now let  $K \in \mathbb{N}$  and  $K \gg 1$ . For  $j \in \mathbb{N}$ , define  $h_j(x) := \chi_{\{j \leq |x| \leq j+j^{-1/2}\}}(x)$ , and let  $f(x) := \chi_{\{0 \leq |x| < 1\}}(x) + \sum_{j=1}^K h_j(x)$ . Then

$$\begin{aligned} \|f\|_{\mathcal{M}_{q_1}^{p_1}} &\geq |B(0, K + K^{-1/2})|^{\frac{1}{q_1} - \frac{1}{p_1}} \left( \int_{B(0, K + K^{-1/2})} |f(x)|^{p_1} dx \right)^{1/p_1} \\ &\geq C(K + K^{-1/2})^{\frac{d}{q_1} - \frac{d}{p_1}} (K + K^{-1/2})^{\frac{d}{p_1} - \frac{d}{2p_1}} = C(K + K^{-1/2})^{\frac{d}{q_1} - \frac{1}{2p_1}}. \end{aligned} \quad (4.1)$$

Meanwhile,

$$\begin{aligned} \|f\|_{\mathcal{M}_{q_2}^{p_2}} &= |B(0, K + K^{-1/2})|^{\frac{1}{q_2} - \frac{1}{p_2}} \left( |B(0, 1)| + \sum_{j=1}^K |\{x \in B(0, K + K^{-1/2}) : j \leq |x| \leq j + j^{-1/2}\}| \right)^{\frac{1}{p_2}} \\ &\leq C(K + K^{-1/2})^{\frac{d}{q_2} - \frac{1}{2p_2}}. \end{aligned} \quad (4.2)$$

By combining (4.1), (4.2),  $q_1 = q_2$ , and  $\|f\|_{\mathcal{M}_{q_1}^{p_1}} \leq C\|f\|_{\mathcal{M}_{q_2}^{p_2}}$ , we get

$$(K + K^{-1/2})^{\frac{1}{2p_2} - \frac{1}{2p_1}} \leq C.$$

As this holds for every  $K \gg 1$ , we conclude that  $p_1 \leq p_2$ .  $\square$

*Proof of Theorem 1.5 (ii).* By arguing as in the proof of Theorem 1.5 (i) and using the identities  $\|\chi_{B(0,r)}\|_{w\mathcal{M}_{q_1}^{p_1}} = |B(0,r)|^{1/q_1}$  and  $\|\chi_{B(0,r)}\|_{w\mathcal{M}_{q_2}^{p_2}} = |B(0,r)|^{1/q_2}$ , we have  $q_1 = q_2$ . Assume to the contrary that  $p_1 > p_2$ . Define  $f$  by (2.2). By a similar argument as in the proof of Theorem 1.3 (ii)-(iii), we have  $f \in w\mathcal{M}_{q_2}^{p_2}$  but  $f \notin w\mathcal{M}_{q_1}^{p_1}$ , which contradicts  $w\mathcal{M}_{q_2}^{p_2} \subseteq w\mathcal{M}_{q_1}^{p_1}$ . Hence  $p_1 \leq p_2$ .  $\square$

*Proof of Theorem 1.5 (iii).* Since  $\mathcal{M}_{q_2}^{p_2} \subseteq w\mathcal{M}_{q_2}^{p_2}$ , we have  $\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$ . Therefore, by virtue of Theorem 1.5 (ii), we have  $q_1 = q_2$  and  $p_1 \leq p_2$ . Now, assume to the contrary that  $p_1 = p_2$ . According to Theorem 1.4, there exists  $f_0 \in w\mathcal{M}_{q_2}^{p_2}$  such that  $f_0 \notin \mathcal{M}_{q_1}^{p_1}$ . This contradicts  $w\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$ . Thus  $p_1 < p_2$ , as desired.  $\square$

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